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A XIN-AIYAMA TYPE THEOREM FOR SPACELIKE HYPERSURFACES IN $-\mathbb{R}\times\mathbb{H}^n$

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Abstract

In this paper, by applying the technique developed by Rosenberg in [11] to establish sharp height estimates in Riemannian space forms, we obtain a Xin-Aiyama type theorem concerning complete space-like hypersurfaces immersed with constant mean curvature in the Lorentzian product space $-\mathbb{R}\times\mathbb{H}^n$. In particular, we treat the case when the spacelike hypersurface is a vertical graph in such spacetime.

1. Introduction

Interest in the study of spacelike hypersurfaces in Lorentzian manifolds has increased very much in recent years, from both the physical and mathematical points of view. For example, it was pointed out by Marsdan and Tipler in [10] and Stumbles in [12] that space-like hypersurfaces with constant mean curvature in arbitrary spacetime play

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an important part in the relativity theory. They are convenient as initial hypersurfaces for the Cauchy problem in arbitrary spacetime and for studying the propagation of gravitational radiation.

From a mathematical point of view, that interest is also motivated by the fact that these hypersurfaces exhibit nice Bernstein-type properties. Actually, Calabi in [6], for $n \leq 4$, and Cheng and Yau in [9], for arbitrary n, showed that the only complete immersed spacelike hypersurfaces of the $(n+1)$-dimensional Lorentz-Minkowski space $\mathbb{L}^{n+1}$ with zero mean curvature are the spacelike hyperplanes.

More recently, Xin in [13] and Aiyama in [1] simultaneously and independently characterized the spacelike hyperplanes as the only complete constant mean curvature spacelike hypersurfaces immersed in $\mathbb{L}^{n+1}$, and having the image of its Gauss map contained into a geodesic ball of the n-dimensional hyperbolic space $\mathbb{H}^n$.

In this paper, by applying the technique developed by Rosenberg in [11] to establish sharp height estimates in Riemannian space forms, we study the complete spacelike hypersurfaces immersed with constant mean curvature in the Lorentzian product $-\mathbb{R} \times \mathbb{H}^n$. We obtain the result below (cf. Theorem 3.2).

Let $\psi : \sum^n \to -\mathbb{R} \times \mathbb{H}^n$ *be a complete spacelike hypersurface with one end. Suppose that the mean curvature H is constant. If the height function h of* $\sum^n$ *is such that* $\|\nabla h\|^2 \leq \alpha H^2$, *for some constant* $0 < \alpha < \dfrac{n}{n-1}$, *then its end is not divergent.*

We want to point out that, when the ambient spacetime is the Lorentz-Minkowski space $\mathbb{L}^{n+1} = -\mathbb{R} \times \mathbb{R}^n$, the hypothesis of the boundedness of the norm of the gradient of the height function is equivalent to the hypothesis of the hyperbolic image of the spacelike hypersurface be contained into a geodesic ball of the hyperbolic space (cf. Remark 3.3). Consequently, this our previous theorem can be seen as a version for $-\mathbb{R} \times \mathbb{H}^n$ of the theorems of Xin and Aiyama.

In the case that the spacelike hypersurface is a vertical graph, we obtain the following (cf. Theorem 4.2).

Let $\sum^n(u)$ *be a complete spacelike vertical graph with one end in* $-\mathbb{R}\times\mathbb{H}^n$. *Suppose that the mean curvature H is constant. If the function u is such that* $|Du|^2 \leq \frac{\alpha H^2}{1+\alpha H^2}$, *for some constant* $0 < \alpha < \frac{n}{n-1}$, *then the end of* $\sum^n(u)$ *is not divergent.*

We observe that Albujer in [2] obtained new explicit examples of complete and non-complete entire maximal (that is, with zero mean curvature) graphs in $-\mathbb{R}\times\mathbb{H}^n$. Moreover, Albujer and Alías gave in [3] a Calabi-Bernstein theorem for maximal surfaces immersed into the Lorentzian product space $-\mathbb{R}\times M^2$, where M^2 is a connected Riemannian surface of nonnegative Gaussian curvature.

Finally, we also note that in [7], Caminha jointly with the author have studied complete vertical graphs of constant mean curvature in the hyperbolic and steady state spaces. Under appropriate restrictions on the values of the mean curvature and the growth of the height function, they obtained necessary conditions for the existence of such a graph. In the 2-dimensional case, they applied their analytical framework to prove Bernstein-type results in each of these ambient spaces.

2. Preliminaries

In what follows, we deal with a spacelike hypersurface $\sum^n$ immersed into a $(n+1)$-dimensional Lorentzian product space $\overline{M}^{n+1}$ of the form $\mathbb{R}\times M^n$, where M^n is an n-dimensional connect Riemannian manifold and $\overline{M}^{n+1}$ is endowed with the Lorentzian metric

$$\langle , \rangle = -\pi^*_{\mathbb{R}}(dt^2) + \pi^*_M(\langle , \rangle_M),$$

where $\pi_{\mathbb{R}}$ and π_M denote the canonical projections from $\mathbb{R} \times M$ onto each factor, and $\langle , \rangle_M$ is the Riemannian metric on M^n. For simplicity, we will just write $\overline{M}^{n+1} = -\mathbb{R} \times M^n$ and $\langle , \rangle = -dt^2 + \langle , \rangle_M$. In this setting, for a fixed $t_0 \in \mathbb{R}$, we say that $M^n_{t_0} = \{t_0\} \times M^n$ is a *slice* of $\overline{M}^{n+1}$.

A smooth immersion $\psi : \sum^n \to -\mathbb{R} \times M^n$ of an n-dimensional connected manifold $\sum^n$ is said to be a *spacelike hypersurface*, if the induced metric via ψ is a Riemannian metric on $\sum^n$, which as usual, is also denoted for $\langle , \rangle$. In that case, since

$$\partial_t = (\partial / \partial_t)_{(t,\, x)}, \quad (t,\, x) \in -\mathbb{R} \times M^n,$$

is a unitary timelike vector field globally defined on the ambient spacetime, then there exists a unique timelike unitary normal field N globally defined on the spacelike hypersurface $\sum^n$, which is in the same time-orientation as ∂_t, so that

$$\langle N,\, \partial_t \rangle \leq -1 < 0 \text{ on } \sum\nolimits^n.$$

We will refer to that normal field N as the future-pointing Gauss map of the spacelike hypersurface $\sum^n$. Its opposite will be referred as the past-pointing Gauss map of $\sum^n$.

Let $\overline{\nabla}$ and ∇ denote the Levi-Civita connections in $-\mathbb{R} \times M^n$ and $\sum^n$, respectively. Then, the Gauss and Weingarten formulae for the spacelike hypersurface $\psi : \sum^n \to -\mathbb{R} \times M^n$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y - \langle AX,\, Y \rangle N,$$

and

$$AX = -\overline{\nabla}_X N,$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\sum)$. Here, $A : \mathfrak{X}(\sum) \to \mathfrak{X}(\sum)$ stands for the shape operator (or Weingarten endomorphism) of $\sum^n$ with respect to either the future or the past-pointing Gauss map N. It is well know, A defines a self-adjoint linear operator on each tangent space $T_p \sum$, and its eigenvalues $\kappa_1(p), \cdots, \kappa_n(p)$ are the principal curvatures of $\sum^n$ at p. Associated to the shape operator A, there are n algebraic invariants given by

$$S_r(p) = \sigma_r(\kappa_1(p), \cdots, \kappa_n(p)), \quad 1 \leq r \leq n,$$

where $\sigma_r : \mathbb{R}^n \to \mathbb{R}$ is the elementary symmetric function in $\mathbb{R}^n$ given by

$$\sigma_r(x_1, \cdots, x_n) = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}.$$

Observe that, the characteristic polynomial of A can be written in terms of the $S_r's$ as

$$\det(tI - A) = \sum_{r=0}^{n} (-1)^r S_r t^{n-r},$$

where $S_0 = 1$ by definition. The r-mean curvature H_r of the spacelike hypersurface $\sum^n$ is then defined by

$$\binom{n}{r} H_r = (-1)^r S_r(\kappa_1, \cdots, \kappa_n) = S_r(-\kappa_1, \cdots, -\kappa_n).$$

In particular, when $r = 1$,

$$H_1 = -\frac{1}{n} \sum_{i=1}^{n} \kappa_i = -\frac{1}{n} \operatorname{tr}(A) = H,$$

is the mean curvature of $\sum^n$, which is the main extrinsic curvature of the hypersurface. The choice of the sign $(-1)^r$ in our definition of H_r is motivated by the fact that in that case, the mean curvature vector is given by $\vec{H} = HN$. Therefore, $H(p) > 0$ at a point $p \in \sum^n$, if and only if $\vec{H}(p)$ is in the same time-orientation as $N(p)$ (in the sense that $\langle \vec{H}, N \rangle_p < 0$).

When $r = 2$, H_2 defines a geometric quantity, which is related to the (intrinsic) scalar curvature R of the hypersurface. For instance, when the ambient spacetime $\overline{M}^{n+1}$ has constant sectional curvature $\overline{\kappa}$, it follows from the Gauss's equation that

$$R = n(n-1)(\overline{\kappa} - H_2).$$

Moreover, in the 3-dimensional case, denoting by $K_{\sum}$, the Gaussian curvature of the spacelike surface $\psi : \sum^2 \to \overline{M}^3$, we have that

$$K_{\sum} = \overline{\kappa} - H_2.$$

Now, we consider two particular functions naturally attached to a spacelike hypersurface $\sum^n$ immersed into a Lorentzian product space $-\mathbb{R} \times M^n$: the (vertical) height function $h = (\pi_{\mathbb{R}})|_{\sum}$ and the support function $\langle N, \partial_t \rangle$, where N denotes the Gauss map of $\sum^n$ and ∂_t is the coordinate vector field induced by the universal time on $-\mathbb{R} \times M^n$.

We observe that the gradient of $\pi_{\mathbb{R}}$ on $-\mathbb{R} \times M^n$ is

$$\overline{\nabla}\pi_{\mathbb{R}} = -\langle \overline{\nabla}\pi_{\mathbb{R}}, \partial_t \rangle = -\partial_t,$$

so that, the gradient of h on $\sum^n$ is

$$\nabla h = (\overline{\nabla}\pi_{\mathbb{R}})^{\top} = -\partial_t^{\top}.$$

Throughout this paper, for a given vector field Z along the immersion, we will denote by $Z^{\top} \in \mathfrak{X}(\sum)$ its tangential component, that is,

$$Z = Z^{\top} - \langle N, Z \rangle N.$$

In particular,

$$\nabla h = -\partial_t - \langle N, \partial_t \rangle N,$$

and we easily get

$$\|\nabla h\|^2 = \langle N, \partial_t \rangle^2 - 1,$$

where $\| . \|$ denotes the norm of a vector field on $\sum^n$. Since ∂_t is parallel on $-\mathbb{R} \times M^n$, we have that

$$\overline{\nabla}_X \partial_t = 0,$$

for every tangent vector field $X \in \mathfrak{X}(\sum)$. Writing $\partial_t = -\nabla h - \langle N, \partial_t \rangle N$ along the hypersurface $\sum^n$ and by using Gauss and Weingarten formulae, we get that

$$\nabla_X \nabla h = \langle N, \partial_t \rangle AX,$$

for every tangent vector field $X \in \mathfrak{X}(\sum)$. Therefore, we obtain the Laplacian on $\sum^n$ of the height function h.

Lemma 2.1. *Let* $\psi : \sum^n \to -\mathbb{R} \times M^n$ *be a spacelike hypersurface with Gauss map N. Then,*

$$\Delta h = -nH \langle N, \partial_t \rangle.$$

Moreover, as a particular case of Corollary 8.2 in [5], we also obtain the Laplacian on $\sum^n$ of the support function $\langle N, \partial_t \rangle$.

Lemma 2.2. *Let* $\psi : \sum^n \to -\mathbb{R} \times M^n$ *be a spacelike hypersurface with Gauss map N. Suppose that the Riemannian fiber* M^n *has constant sectional curvature* κ *and that the mean curvature H is constant. Then,*

$$\Delta\langle N, \partial_t \rangle = (n^2H^2 - n(n-1)H_2 + (n-1)\kappa\|\nabla h\|^2)\langle N, \partial_t \rangle.$$

Remark 2.3. The formulae collected in the above lemmas are the Lorentzian versions of the ones obtained by Cheng and Rosenberg (cf. [8], Lemmas 4.1 and 4.2). We also note that, Alías and Colares obtained a generalization of these formulae in the context of the generalized Robertson Walker spacetimes (cf. [5], Lemma 4.1 and Corollary 8.5). Moreover, Albujer and Alías obtained in [3], the formulae of the previous lemmas for the case of a spacelike surface immersed in a three-dimensional Lorentzian product space. For an alternative proof of the above formulae, we suggest [7].

3. Complete Hypersurfaces with One End in $-\mathbb{R} \times \mathbb{H}^n$

In what follows, we will consider the upper half-space model for the n-dimensional hyperbolic space, that is,

$$\mathbb{H}^n = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n;\ x_n > 0\},$$

endowed with the complete metric

$$\langle , \rangle_{\mathbb{H}^n} = \frac{1}{x_n^2}(dx_1^2 + \cdots + dx_n^2).$$

In this setting, we deal with complete spacelike hypersurface $\psi : \sum^n \to -\mathbb{R} \times \mathbb{H}^n$ with *one end* $\mathcal{C}^n$, that is, a spacelike hypersurface $\sum^n$ that we can regard as

$$\sum^n = \sum_t^n \cup \mathcal{C}^n,$$

where $\sum_t^n$ is a compact hypersurface with boundary contained into a slice $\mathbb{H}_t^n = \{t\} \times \mathbb{H}^n$ and $\mathcal{C}^n$ is diffeomorphic to the cylinder $[t, \infty) \times \mathbb{S}^{n-1}$.

Definition 3.1. Let $\sum^n = \sum_t^n \cup \mathcal{C}^n$ be a spacelike hypersurface with one end in $-\mathbb{R} \times \mathbb{H}^n$. We say that the end of $\sum^n$ is *divergent*, if considering $\mathcal{C}^n$ with coordinates $p = (s, q) \in [t, \infty) \times \mathbb{S}^{n-1}$, we have that

$$\lim_{s \to \infty} h_{\mathcal{C}}(p) = \infty,$$

where $h_{\mathcal{C}}$ denotes the height function of the end $\mathcal{C}^n$.

Now, we present our main result.

Theorem 3.2. *Let* $\psi : \sum^n \to -\mathbb{R} \times \mathbb{H}^n$ *be a complete spacelike hypersurface with one end. Suppose that the mean curvature* H *is constant. If the height function* h *of* $\sum^n$ *is such that* $\|\nabla h\|^2 \leq \alpha H^2$, *for some constant* $0 < \alpha < \frac{n}{n-1}$, *then its end is not divergent.*

Proof. Let $\sum^n = \sum_t^n \cup \mathcal{C}^n$ be a complete spacelike hypersurface with one end $\mathcal{C}^n$. Since the case $H = 0$ is trivial, we will suppose, after an appropriated choice of the Gauss map N of $\sum^n$, that $H > 0$. We then, consider the two possible cases:

(a) Suppose that N is future-pointing.

In this case, we define on $\sum$ the function

$$\varphi = ch_{\sum_t} - \langle N, \partial_t \rangle,$$

where $h_{\sum_t}$ denotes the height function of $\sum_t^n$ and c is a negative constant that we will choose appropriately. Since $|\langle N, \partial_t \rangle| \geq 1$, we have that

$$\varphi_{|\partial\Sigma_t} \leq \max_{\partial\Sigma_t} |\langle N, \partial_t \rangle| \leq \max_{\Sigma} \langle N, \partial_t \rangle^2.$$

Furthermore, from Lemmas 2.1 and 2.2,

$$\Delta\varphi = -\langle N, \partial_t \rangle (nH(c + H) - n(n-1)\|\nabla h_{\Sigma_t}\|^2 + n(n-1)(H^2 - H_2)).$$

On the other hand, by the Cauchy-Schwartz inequality,

$$H^2 - H_2 \geq 0.$$

Thus, we obtain that

$$\Delta\varphi \geq -\langle N, \partial_t \rangle (nH(c + H) - n(n-1)\|\nabla h_{\Sigma_t}\|^2).$$

Consequently, considering our assumption that $\|\nabla h\|^2 \leq \alpha H^2$ with $0 \leq \alpha < \frac{n}{n-1}$, by taking

$$0 > c \geq \left(\left(\frac{n-1}{n}\right)\alpha - 1\right) H$$

in the definition of the function φ, we get that $\Delta\varphi \geq 0$ on $\sum_t^n$. Then, since $\|\nabla h\|^2 = \langle N, \partial_t \rangle^2 - 1$, we conclude from the maximum principle that

$$\varphi \leq \max_{\Sigma} \langle N, \partial_t \rangle^2 \leq \max_{\Sigma} \|\nabla h\|^2 + 1.$$

Thus,

$$|h_{\Sigma_t}| \leq \frac{n\alpha H}{n - \alpha(n-1)},$$

with $0 \leq \alpha < \frac{n}{n-1}$. Therefore, since this estimate of h_{Σ_t} does not depend on the parameter t, we conclude that the end $\mathcal{C}^n$ of $\sum^n$ is not divergent.

(b) Suppose that N is past-pointing.

In this another case, we define on $\sum_t^n$ the function

$$\varphi = ch_{\sum_t} + \langle N, \partial_t \rangle,$$

where now c is a positive constant. From this point, by taking

$$0 < c \leq \left(1 - \left(\frac{n-1}{n}\right)\alpha\right) H,$$

the proof carries in a similar way of the previous case, and we again conclude that the end $\mathcal{C}^n$ of $\sum^n$ is not divergent. ■

Remark 3.3. We note that, when $\sum^n$ is an immersed spacelike hypersurface of the Lorentz-Minkowski space $\mathbb{L}^{n+1} = -\mathbb{R} \times \mathbb{R}^n$, the timelike unit normal vector field $N \in \mathfrak{X}^{\perp}(\sum)$ can be regarded as a map $N : \sum^n \to \mathbb{H}^n$. Here, we are considering the Minkowski model of the n-dimensional hyperbolic space $\mathbb{H}^n$, that is,

$$\mathbb{H}^n = \{x \in \mathbb{L}^{n+1}; \langle x, x \rangle = -1, x_1 \geq 1\}.$$

In this setting, the image $N(\sum)$ is called the *hyperbolic image* of $\sum^n$. On the other hand, given a geodesic ball $B(a, \varrho)$ in $\mathbb{H}^n$ of radius $\varrho > 0$ centered at a point $a \in \mathbb{H}^n$, we recall that $B(a, \varrho)$ is characterized as the following

$$B(a, \varrho) = \{p \in \mathbb{H}^n; -\cosh \varrho \leq \langle p, a \rangle \leq -1\}.$$

Consequently, if the hyperbolic image of $\sum^n$ is contained into a geodesic ball of radius ϱ, then (subtending the compositions with the isometry between the Lorentzian product and the canonical models of $\mathbb{L}^{n+1}$), we have that

$$1 \leq |\langle N, \partial_t \rangle| \leq \cosh \varrho.$$

Therefore, since $\|\nabla h\|^2 = \langle N, \partial_t \rangle^2 - 1$, when the ambient spacetime is the Lorentz-Minkowski space, the hypothesis of the boundedness of the norm of the gradient of the height function is equivalent to the hypothesis of the hyperbolic image of the spacelike hypersurface be contained into a geodesic ball of the hyperbolic space. In this sense, the Theorem 3.2 can be seen as a version for $-\mathbb{R} \times \mathbb{H}^n$ of the theorems of Xin in [13] and Aiyama in [1].

If the height function $h : \sum^n \to \mathbb{R}$ is such that $h \geq t_0$ for some $t_0 \in \mathbb{R}$, then we say that $\sum^n$ is a spacelike hypersurface over the slice $\mathbb{H}^n_{t_0} = \{t_0\} \times \mathbb{H}^n$. In this setting, by being contained in a slab $[t_0, t_1] \times \mathbb{H}^n$, we mean between the slices $\mathbb{H}^n_{t_0}$ and $\mathbb{H}^n_{t_1}$, with $t_0 < t_1$. With such notations and conventions, one can then reason as in the previous result to obtain the following height estimate.

Theorem 3.4. *Let* $\psi : \sum^n \to -\mathbb{R} \times \mathbb{H}^n$ *be a complete spacelike hypersurface with one end over a slice* $\mathbb{H}^n_{t_0}$, $t_0 \in \mathbb{R}$. *Suppose that the mean curvature* H *is constant. If the height function* h *of* $\sum^n$ *is such that* $\|\nabla h\|^2 \leq \alpha H^2$, *for some constant* $0 < \alpha < \dfrac{n}{n-1}$, then $\sum^n$ *is contained in the slab* $[t_0, t_1] \times \mathbb{H}^n$, *where* $t_1 = t_0 + \dfrac{n\alpha H}{n - \alpha(n-1)}$.

Remark 3.5. In [4], Alías and Dajczer studied complete properly immersed surfaces contained in a slab of a Riemannian warped product $\mathbb{R} \times_\varrho \mathbb{P}^2$, where $\mathbb{P}^2$ is complete with nonnegative Gaussian curvature. Under certain restrictions on the mean curvature of the surface, they showed that such an immersion does not exist or must be a slice $\{t\} \times \mathbb{P}^2$.

4. Complete Vertical Graphs in $-\mathbb{R} \times \mathbb{H}^n$

Let $\Omega \subseteq \mathbb{H}^n$ be a connected domain of the n-dimensional hyperbolic space. A *vertical graph* over Ω is determined by a smooth function $u \in C^{\infty}(\Omega)$ and it is given by

$$\sum\nolimits^n(u) = \{(u(x),\, x);\ x \in \Omega\} \subset -\mathbb{R} \times \mathbb{H}^n.$$

The metric induced on Ω from the Lorentzian metric on the ambient space via $\sum^n(u)$ is

$$\langle , \rangle = -du^2 + \langle , \rangle_{\mathbb{H}^n}.$$

The graph is said to be *entire*, if $\Omega = \mathbb{H}^n$. We note that this occurs when $\sum^n(u)$ is complete. It can be easily seen that a graph $\sum^n(u)$ is a spacelike hypersurface, if and only if $|Du|^2 < 1$ being Du, the gradient of u in Ω and $|Du|$ its norm, both with respect to the hyperbolic metric $\langle , \rangle_{\mathbb{H}^n}$ in Ω.

If $\sum^n(u)$ is a spacelike vertical graph over a domain, then the vector field

$$N(x) = \frac{1}{\sqrt{1 - |Du|^2}}\,(\partial_t|_{(u(x),\, x)} + Du(x)), \quad x \in \Omega,$$

defines the future-pointing Gauss map of $\sum^n(u)$. Moreover, the shape operator A of $\sum^n(u)$ with respect to N is given by

$$AX = -\frac{1}{\sqrt{1 - |Du|^2}}\, D_X Du - \frac{\langle D_X Du,\, Du\rangle_{\mathbb{H}^n}}{(1 - |Du|^2)^{3/2}}\, Du,$$

for every tangent vector field X on Ω, where D denotes the Levi-Civita connection in Ω with respect to the metric $\langle , \rangle_{\mathbb{H}^n}$. It follows from here, that the mean curvature of a spacelike graph $\sum^n(u)$, $H(u) = -\frac{1}{2}\operatorname{tr}(A)$, is given by

$$2H(u) = \operatorname{Div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right),$$

where Div stands for the divergence operator on Ω with respect to the metric $\langle , \rangle_{\mathbb{H}^n}$. In particular, $\sum^n(u)$ is a maximal (that is, with zero mean curvature) graph, the function u satisfies the following partial differential equation on the domain Ω,

$$\operatorname{Div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = 0, \quad |Du|^2 < 1.$$

Remark 4.1. In [2], Albujer obtained new explicit examples of complete and non-complete entire maximal graphs in $-\mathbb{R} \times \mathbb{H}^n$. The existence of these entire maximal graphs shows that entire maximal graphs in this Lorentzian product space are not necessarily complete, on the contrary that in the Lorentz-Minkowski space. On the other hand, Albujer and Alías gave in [3], a Calabi-Bernstein theorem for maximal surfaces immersed into the Lorentzian product space $-\mathbb{R} \times M^2$, where M^2 is a connected Riemannian surface of nonnegative Gaussian curvature.

Theorem 4.2. *Let $\sum^n(u)$ be a complete spacelike vertical graph with one end in $-\mathbb{R} \times \mathbb{H}^n$. Suppose that the mean curvature H is constant. If the function u is such that $|Du|^2 \leq \frac{\alpha H^2}{1+\alpha H^2}$, for some constant $0 < \alpha < \frac{n}{n-1}$, then the end of $\sum^n(u)$ is not divergent.*

Proof. We have that

$$N = -\langle N, \partial_t \rangle \partial_t + (\pi_{\mathbb{H}^n})_*(N).$$

Consequently,

$$((\pi_{\mathbb{H}^n})_*(N))^{\top} = -\langle N, \partial_t \rangle \nabla h,$$

and

$$\|\nabla h\|^2 = \langle (\pi_{\mathbb{H}^n})_*(N), (\pi_{\mathbb{H}^n})_*(N) \rangle_{\mathbb{H}^n}.$$

Thus, since

$$N = \frac{1}{\sqrt{1 - |Du|^2}} (\partial_t + Du),$$

we get

$$\|\nabla h\|^2 = \frac{|Du|^2}{1 - |Du|^2}.$$

Therefore, the assumption that $|Du|^2 \leq \frac{\alpha H^2}{1 + \alpha H^2}$, for some constant $0 \leq \alpha < \frac{n}{n-1}$, guarantees that $\|\nabla h\|^2 \leq \alpha H^2$, for some constant $0 \leq \alpha < \frac{n}{n-1}$, and the results follows from Theorem 3.2. ■

Finally, we state the analogue of the Theorem 3.4 for the case of a complete vertical graph.

Theorem 4.3. *Let* $\sum^n(u)$ *be a complete spacelike vertical graph with one end in* $-\mathbb{R} \times \mathbb{H}^n$ *and over a slice* $\mathbb{H}^n_{t_0}$, $t_0 \in \mathbb{R}$. *Suppose that the mean curvature H is constant. If the function u is such that* $|Du|^2 \leq \frac{\alpha H^2}{1 + \alpha H^2}$, *for some constant* $0 < \alpha < \frac{n}{n-1}$, *then* $\sum^n(u)$ *is contained in the slab* $[t_0, t_1] \times \mathbb{H}^n$, *where* $t_1 = t_0 + \frac{n\alpha H}{n - \alpha(n-1)}$.

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